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LETTER TO THE EDITOR

An explicit relationship between steady-state size distribution and breakage kernel for limited breakage processes

M Kostoglou and A J Karabelas

Chemical Process Engineering Research Institute and Department of Chemical Engineering, University Box 455, Aristotle University of Thessaloniki, GR 54006 Thessaloniki, Greece

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Abstract. The steady-state particle-size distribution is examined, resulting from a breakage process with a maximum stable size. If the latter is much smaller than the characteristic size of the initial distribution, the steady-state distribution for *continuous* breakage kernels is independent of the breakage frequency and of initial conditions and is shown to be a simple function of the breakage kernel. For *discontinuous* kernels, the steady-state size distribution is always dependent on the initial conditions.

1. Introduction

Breakage of particulate matter into smaller fragments is encountered in many natural phenomena and technological processes. For instance at the geological time scale, a fragmentation process is apparently responsible for the distribution of particulate material on the earth surface. Technological aspects of fragmentation concern mineral processing, polymer degradation and break-up of liquid droplets or air bubbles. A rather extensive account for processes based on fragmentation is given in the review by Redner [1]. The term breakage is considered synonymous to fragmentation and used mainly in fluid processes while fragmentation refers to solids.

It was already recognized in the early studies on fragmentation that a critical particle size X_m may exist below which there is no further break-up. This is widely accepted for certain systems such as the turbulent flow of liquid–liquid dispersions [2]; indeed it is considered that the turbulent flow field cannot cause breakage of droplets below a certain size related to the turbulent eddy structure. On the other hand, forces acting on large droplets (of size greater than X_m) can lead to daughter particles much smaller than X_m or even of the smallest eddy of the turbulent flow. One may call this model of limited breakage, *type-I breakage*. There is another fundamentally different model of limited breakage; it is relevant to solid and polymer processes and is characterized by a critical size below which particles cannot exist in the system. This model of limited breakage may be called *type-II breakage* [3]. It is clear that type-I breakage imposes a minimum size on parent particles while type-II breakage imposes a minimum size on daughter particles. It is noted that the small cut-off size given by Redner [1] does not actually lead to a limited breakage mechanism because that restriction is on the ratio of daughter to parent particles and not on the absolute size of either one of them. An obvious difference between continuous and limited breakage is that

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in the former the PSD keeps changing with time, while in the latter the process results in a steady state.

The existence of a critical size for type-I breakage implies that the particle-size distribution reaches a steady state where all the particles are smaller than this critical size. If only breakage takes place, this steady state may be called *static*, in the sense that particles smaller than the critical size remain unchanged, and additionally it may depend in principle on the earlier states of the system. By contrast, a *dynamic* steady state resulting from a combination of two or more competing mechanisms, e.g. breakage and coalescence [4]. is characterized by a ceaseless alteration of particles. The dynamic steady state is independent of earlier states of the system and thus of the initial conditions.

Although there is an extensive literature on the evaluation of the critical size for the type-I limited breakage as a function of the physicochemical characteristics of the system, there are only a few solutions, mainly numerical, of the mathematical problem [5, 6]. On the other hand analytical solutions for the type-II limited breakage have been recently given [3] for the case of binary random breakage using the method of Laplace transform. The object of this work is to study analytically the steady-state PSD for type-I limited breakage.

2. Mathematical formulation

The evolution of size distribution of dispersed particles for type-I limited breakage is given by

$$\frac{\mathrm{d}f'(x',t)}{\mathrm{d}t} = \int_{x'}^{\infty} v'(y')p'(x',y')b'(y')f'(y',t)\,\mathrm{d}y' - b'(x')f'(x',t) \qquad x' > X_m \tag{1a}$$

$$\frac{\mathrm{d}f'(x',t)}{\mathrm{d}t} = \int_{X_m}^{\infty} v'(y')p'(x',y')b'(y')f'(y',t)\,\mathrm{d}y' \qquad x' < X_m \tag{1b}$$

where the symbols involved have the following meaning: t time, x' particle volume, $f(x',t)$ particle number density distribution, $b'(x')$ breakage probability of particles of volume x', $v'(y')p'(x',y')$ distribution of particles of volume x' resulting from the break-up of a particle of volume y', $v'(y')$ number of particles resulting from the break-up of a particle of volume y'.

Let $f'_0(x') = f'(x',0)$ be the initial distribution. The total volume concentration, the total number concentration and the mean size of the initial distribution are, respectively:

$$M = \int_0^{\infty} x f'_0(x)\,\mathrm{d}x$$

$$N_0 = \int_0^{\infty} f'_0(x)\,\mathrm{d}x$$

$$x_0 = \frac{M}{N_0}.$$

The functions and variables already introduced can be expressed in dimensionless form, as follows:

$$x = \frac{x'}{x_0} \qquad y = \frac{y'}{x_0} \qquad \tau = b'(x_0)t \qquad x_m = \frac{X_m}{x_0}$$

$$b(x) = \frac{b'(x)}{b'(x_0)} \qquad f(x,\tau) = \frac{x_0 f'(x',t)}{N_0}$$

$$p(x,y) = x_0 p'(x',y') \qquad v(y) = v'(y')$$

and equations 1(a) and (1b) can be written as

$$\frac{\mathrm{d}f(x,\tau)}{\mathrm{d}\tau} = \int_x^\infty v(y)p(x,y)b(y)f(y,\tau)\,\mathrm{d}y - b(x)f(x,\tau) \qquad x > x_m \tag{2a}$$

$$\frac{\mathrm{d}f(x,\tau)}{\mathrm{d}\tau} = \int_{x_m}^\infty v(y)p(x,y)b(y)f(y,\tau)\,\mathrm{d}y \qquad x < x_m. \tag{2b}$$

There is a large number of solutions to the above problem available for the case $x_m = 0$. For certain simple forms of functions $b(x)$ and $v(y)p(x,y)$ analytical solutions exist [7, 8]. Similarity tranformations [9] can be used if the above functions satisfy certain requirements. Finally, for a general form of kernels, there are specialized numerical methods [10] and Monte Carlo simulations [11]. Because of the linearity of the problem the solution for an arbitrary initial distribution can be obtained from the superposition of solutions for monodisperse initial distributions. The general solution to the above problem for monodisperse initial distribution is:

for $x < x_m$

$$f(x,\tau) = \sum_{i=0}^\infty A^{(i)}(x)\frac{\tau^{i+1}}{(i+1)!} + \delta(x-1)\mathrm{e}^{-\tau} \tag{3a}$$

for $x > x_m$

$$f(x,\tau) = \sum_{i=0}^\infty \int_{x_m}^1 A(x,y)A^{(i)}(y)\,\mathrm{d}y\frac{\tau^{i+2}}{(i+2)!} + A(x,1)\mathrm{e}^{-\tau} \tag{3b}$$

where

$$A^{i+1}(x) = \int_x^1 A(x,y)A^{(i)}(y)\,\mathrm{d}y + (1-b(x))A^{(i)}(x) \qquad i = 0, 1, 2 \ldots \infty$$
$$A(x,y) = A^{(0)}(x,y) = b(y)v(y)p(x,y). \tag{3c}$$

3. Solutions for the steady state

The steady-state distribution can be obtained from the above series by substituting $\tau = \infty$. Since this method is quite impractical computationally, one may proceed in a different way.

The following function is introduced:

$$L(x) = b(x)\int_0^\infty f(x,\tau)\,\mathrm{d}\tau \tag{4}$$

which represents the total number of particles with volume x, that suffer breakage during the entire process. This transformation essentially eliminates the breakage frequency.

Integrating equations (2a), and (2b) from $\tau = 0$ to ∞, one obtains:

$$-f_0(x) = \int_x^\infty v(y)p(x,y)L(y)\,\mathrm{d}y - L(x) \qquad x > x_m \tag{5a}$$

$$f_s(x) = \int_{x_m}^\infty v(y)p(x,y)L(y)\,\mathrm{d}y + f_0(x) \qquad x < x_m \tag{5b}$$

where $f_s(x)$ is the dimensionless steady-state particle-size distribution. The function $L(x)$ depends on $v(y)p(x,y)$ and $f_0(x)$, whereas the steady-state size distribution $f_s(x)$ depends additionally on x_m.

One may assume that the breakage kernel is independent of the absolute parent-particle size but that depends only on the ratio x/y. Volume conservation considerations imply

that the kernel has the form $v(y)p(x, y) = \varphi(x/y)/y$. This type of kernel has been used extensively in the literature from prototype models for the solution of the breakage equation [8] to empirical expressions for fitting experimental data [12].

Nambiar *et al* [5] solved numerically the dynamic equation and found the steady-state size distribution in the limit of large times. Kostoglou *et al* [6] solved numerically equation (5) to obtain directly the steady-state size distribution. Both works resulted in the conclusion that for $x_m \ll 1$ the steady-state size distribution (called 'limiting') is independent of the initial size distribution $f_0(x)$. Thus, there is a direct relationship between the steady-state size distribution and the breakage kernel. Exploitation of such a relationship is the specific objective of this work. To proceed one should modify the nondimensionalization of particle volume and of steady-state distribution to render it independent of the initial distribution:

$$\bar{x} = \frac{x}{x_m} \qquad \bar{f}(\bar{x}) = \frac{X_m^2 f'(x')}{M} = x_m^2 f(x). \tag{6}$$

Because of the linearity of the problem and the fact that one expects that the 'limiting' steady state is independent of the initial conditions, it is not restrictive to assume that the initial distribution is monodisperse $\delta(x-1)$. Using the above and the new function $q(x) = L(x) - \delta(x-1)$ equations (5) are modified as:

$$\varphi(x) + \int_x^1 \frac{1}{y}\varphi(x/y)q(y)\,\mathrm{d}y - q(x) = 0 \tag{7a}$$

$$\bar{f}_s(\bar{x}) = x_m^2 \int_{x_m}^1 \frac{1}{y}\varphi\left(\frac{\bar{x}}{y}x_m\right) q(y)\,\mathrm{d}y + x_m^2 b(\bar{x}x_m). \tag{7b}$$

One is interested in the limiting steady-state distribution $f_{sl}(\bar{x})$ that can be computed from equation (7*b*) in the limit $x_m \to 0$. A particular solution to the above problem is given in [6] in the form of an infinite series containing derivatives of the kernel of all orders. This solution has obvious restrictions but it is correct for the few kernels for which the series converges. A general solution is given here, valid for all cases where a steady state exists. Using the new variable of integration $z = \frac{\bar{x}}{y}x_m$ in equation (7*b*), the following relation is obtained:

$$\bar{f}_{sl}(\bar{x}) = \lim_{x_m \to 0} x_m^2 \int_0^{\bar{x}} \frac{1}{z}\varphi(z)q\left(\frac{\bar{x}}{z}x_m\right)\mathrm{d}z. \tag{8}$$

To proceed with the above equation the asymptotic behaviour of $q(x)$ as $x \to 0$ must be known. It is interesting that only the asymptotic behaviour of $q(x)$ is required for evaluating the limiting steady state and not the entire function defined in the interval $[0, 1]$. To find the asymptotic behaviour of $q(x)$, the following procedure is adopted. Equation (1) is multiplied by x^s and then integrated with respect to x from $x = 0$ to ∞. After some algebra the following relation results:

$$Q(s) = \frac{\Phi(s)}{1 - \Phi(s)} \tag{9}$$

where

$$Q(s) = \int_0^1 x^s q(x)\,\mathrm{d}x \qquad \Phi(s) = \int_0^1 x^s \varphi(x)\,\mathrm{d}x.$$

The functions $Q(s)$ and $\Phi(s)$ are related to the Mellin transforms of the functions $q(x)$ and $\varphi(x)$ respectively. Function $\Phi(s)$ is a purely monotonic (decreasing) function of s for $s > 0$. It takes the values $v/2$ and 1 at $s = 0$ and 1, and tends to zero as s tends to infinity.

Function $Q(s)$ is by definition ($q(x) > 0$) positive. According to equation (9) $Q(s)$ takes positive values for $s > 1$ and diverges for $s = 1$. For $s < 1$ it takes negative values with no physical meaning. The above behaviour clearly suggests that $q(x)$ for $x \to 0$ behaves asymptotically like the power function x^{-2}. If A is a constant (dependent on $\varphi(x)$) one may substitute for $x \to 0, q(x) = Ax^{-2}$. Although this procedure is not mathematically precise, it is supported by the available exact solutions. Furthermore, similar considerations have been extensively used in the scaling theory of fragmentation [13].

The above statement is not valid for discontinuous kernels. The function $q(x)$ is also discontinuous and in the limit of small x acquires a fractal-like structure with increasingly closely spaced discontinuities. Nevertheless, equation (9) also holds in this case but the asymptotic behaviour is observed only in an integral sense. For example, for the equal-size binary breakage ($\varphi(x) = \delta(x - \frac{1}{2})$) the asymptotic relation is not valid because there exist values of x arbitrarily close to zero where $q(x) = 0$; it is valid, however, for the mean value of $q(x)$ over finite regions of x. The above behaviour of $q(x)$ suggest that a limiting steady state does not exist for discontinuous breakage kernels. In this case the steady-state size distribution depends on the initial distribution. This result is confirmed using the available exact solutions for the steady state [6].

Substituting the asymptotic relation for $q(x)$ in equation (8) and taking the limit, the following relation is obtained:

$$\bar{f}_s(\bar{x}) = \frac{A}{\bar{x}^2} \int_0^{\bar{x}} z\varphi(z)\,\mathrm{d}z. \tag{10}$$

The parameter A can be evaluated, from the requirement of the total volume conservation $\int_0^1 \bar{x}\,\bar{f}_s(\bar{x})\,\mathrm{d}\bar{x} = 1$ as $A = -(\int_0^1 z\ln(z)\varphi(z)\,\mathrm{d}z)^{-1}$ It is worth noting that the steady-state distribution is independent of the breakage rate $b(x)$. The resulting equation (10) is very important for the so-called inversion problem; i.e. from a measured steady-state size distribution $\bar{f}_s(\bar{x})$, it is very easy to determine the governing breakage kernel from the relation:

$$\varphi(z) = \frac{1}{z\bar{f}_s(1)} \frac{\mathrm{d}(z^2\bar{f}_s(z))}{\mathrm{d}z}. \tag{11}$$

This inversion formula is similar to that for determining the bubble-size distribution from a measured chord-length distribution. In both problems the result is obtained from the first derivative of an experimentally determined function. An extensive analysis for the inversion of such problems from noisy experimental data is given in [14].

The simple form of equation (10) allows analytical solutions for a fairly general class of breakage kernels. Two such very simple cases are analysed, which are representative forms of binary breakage kernels. They are a subset of the general kernels for multiple breakage given in [15]. These two forms are simple substitutes for the two widely used kernels (for binary breakage) in practical cases, i.e. the normal distribution kernel and the U-shaped one.

(i) *Product kernel* ($m = 0, 1, 2\ldots$)

$$\varphi(z) = \frac{2(2m+1)!}{(m!)^2} z^m(1-z)^m. \tag{12a}$$

This kernel is uniform for $m = 0$. For other values of m it has a form similar to that of the well known normal distribution-type kernel. The exponent m is related to the inverse of the standard deviation of the normal-shaped kernel. In the limit $m \to \infty$ this kernel tends to the equal-size breakage kernel [16].

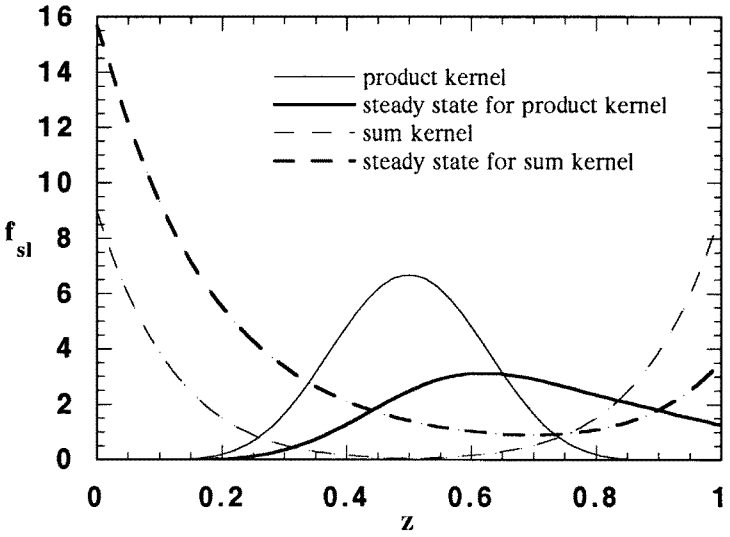

Figure 1. Breakage kernel and 'limiting' steady-state size distribution for product and sum kernels ($m = 8$).

The limiting steady-state size distribution for this kernel is given as:

$$\bar{f}_{sl}(z) = \left(\sum_{i=0}^{i=m} \frac{(-1)^i m!}{i!(m-i)!(m+i+2)^2} \right)^{-1} \sum_{i=0}^{i=m} \frac{(-1)^i m!}{i!(m-i)(m+i+2)} z^{m+i}. \tag{12b}$$

(ii) *Sum kernel* $(m = 1, 2, 3 \ldots)$

$$\varphi(z) = (m+1)[z^m + (1-z)^m]. \tag{13a}$$

This kernel is uniform for $m = 1$. For other values of m it has a U-shaped form. As m increases the kernel tends to represent more 'errosive' behaviour, i.e. preference for daughter particles with very different sizes. The limiting steady-state size distribution for this kernel is given as:

$$\bar{f}_{sl}(z) = \left(\frac{1}{(m+2)^2} + \sum_{i=0}^{m} \frac{(-1)^i m!}{i!(m-i)!(i+2)^2} \right)^{-1} \times \frac{1}{z^2} \left(\frac{x^{m+2} + (1-x)^{m+2}}{m+2} - \frac{(1-x)^{m+1}}{m+1} + \frac{1}{(m+1)(m+2)} \right). \tag{13b}$$

In figure 1 the two kernels (product and sum) and the respective limiting steady-state size distributions are shown for $m = 8$. In general, the steady-state distribution retains the qualitative features of the respective breakage kernel.

4. Conclusion

An explicit relation between the steady-state particle-size distribution and a homogeneous breakage kernel is found for particles undergoing breakage with a maximum stable size much smaller than the initial particle size. The only restriction on the existence of this 'limiting' steady state is that the kernel must be continuous. This relationship greatly simplifies the inverse problem of determining the breakage kernel from an experimentally obtained steady-state distribution.

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